Assignment 4: Surface Integrals

Be sure to show all work, not just the final answer. The assignment is due at the beginning of class, August 18th.

1. Calculate the div and curl of each of the following vector fields:

   (a) $\mathbf{F}(x, y, z) = (x^2 z)\mathbf{i} + \sin(yz)\mathbf{j} + (xyz)\mathbf{k}$

   (b) $\mathbf{F}(x, y, z) = (3x^2 z + ye^x)\mathbf{i} + (e^x)\mathbf{j} + (x^3)\mathbf{k}$

   Is either of these vector fields conservative?

2. Calculate the equation of the tangent plane to the surface with parametric equation

   $$x = uv,\ y = v^2,\ z = ue^u$$

   at $u = 0, v = 1$.

3. Calculate the surface area of the part of the plane $x = z - y$ that is inside the cylinder $y^2 + z^2 = 1$.

4. Evaluate the integral

   $$\int_S \frac{xz}{y}\, dS$$

   where S is the part of the cylinder $x = y^2$ which lies in the first octant between the planes $z = 0$, $z = 5$, $y = 1$, and $y = 4$.

5. Evaluate the integral $\int_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

6. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot dr$, where

   $$\mathbf{F} = (y + \tan\sqrt{x})\mathbf{i} + (e^{y^2} + x^2)\mathbf{j} + (\sin(z^{-1}) + xy)\mathbf{k}$$

   and C is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 2)$.

7. Use Stokes' theorem to calculate

   $$\int_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS$$

   where $\mathbf{F} = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$ and S is the the part of $z = 9 - x^2 - y^2$ where $z \geq 0$.

8. Suppose a cube of volume 1 is submerged in a stream. Use the divergence theorem to calculate the rate of flow of water over the surface of the cube, if the water current is given by the vector field

$$\mathbf{F}(x, y, z) = (xz^2 + y)\mathbf{i} + (xy)\mathbf{j} + (\cos x + x^2)\mathbf{k}$$

(assume that the lower left corner of the cube is located at the point $(0, 0, 0)$).